BINARY OPERATIONS ON SETS

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Reference: *A Gentle Introduction to Group Theory*, Bana Al Subaiei & Muneerah Al Nuwairan, Section 4.1.

A binary operation, denoted by the generic symbol $*$, is an operation defined on a set G that takes two elements of G and maps them onto another element of G. That is, formally

$$*: G\times G\to G \tag{1}$$

The operation must be a function, which means it must satisfy the following conditions:

(1) For each $a,b\in G$, the operation $*$ assigns an image $a*b\in G$.
(2) The assigned image $a*b$ must be in G.
(3) The assigned image must be unique. That is, if $a,b,c,d\in G$ and $a=c$ and $b=d$, then $a*b=c*d$.

The operation is commutative if $a*b=b*a$ for all $a,b\in G$. It is associative if $(a*b)*c=a*(b*c)$ for all $a,b,c\in G$.

Example 1. Let A be the set of negative integers, and $*$ be subtraction, so that $a*b=a-b$. This is not a binary operation, since if $a=-1$ and $b=-2$, we have $a*b=-1-(-2)=+1$, which is not in the set A.

Example 2. Let $A=\mathbb{Q}$, the set of rational numbers, and $*$ is subtraction. In this case, a typical operation is of the form

$$\frac{a}{b}*\frac{c}{d}=\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd} \tag{2}$$

The result on the RHS is also a rational number, so all images of the operation are also in $\mathbb{Q}$.

To test uniqueness, suppose we have $\frac{a}{b}=\frac{1}{2}$, $\frac{c}{d}=\frac{5}{7}$. Then

$$\frac{1}{2}-\frac{5}{7}=\frac{7-10}{14}=-\frac{3}{14} \tag{3}$$

Now suppose we have $\frac{a}{b}=\frac{2}{4}$ and $\frac{c}{d}=\frac{10}{14}$, which are numerically the same as the original rational numbers. We then have

$$\frac{2}{4}-\frac{10}{14}=\frac{28-40}{56}=-\frac{12}{56}=-\frac{3}{14} \tag{4}$$

Thus we get the same result as in 3. In general, if we have, with $m,n\in\mathbb{Z}\backslash\{0\}$

$$\begin{aligned}\frac{na}{nb}*\frac{mc}{md} &= \frac{na}{nb}-\frac{mc}{md} && (5)\\ &= \frac{nmad-nmbc}{nmbd} && (6)\\ &= \frac{ad-bc}{bd} && (7)\end{aligned}$$

Thus the images are unique. Thus this is a binary operation.

Example 3. Let $x,y\in\mathbb{Z}$and $*$ be the operation

$$a*b=xa+yb \tag{8}$$

with $a,b\in\mathbb{R}$ (the real numbers).

We can show that $*$ is associative if and only if $x,y\in\{0,1\}$. If it's associative, then

$$\begin{aligned}a*(b*c) &= xa+y(xb+yc) && (9)\\ &= xa+xyb+y^2c && (10)\\ &= (a*b)*c && (11)\\ &= x(xa+yb)+yc && (12)\\ &= x^2a+xyb+yc && (13)\end{aligned}$$

Thus we must have

$$xa+xyb+y^2c=x^2a+xyb+yc \tag{14}$$

Since a,b,c are arbitrary, we can equate coefficients, giving

$$\begin{aligned}x&=x^2\\ xy&=xy\\ y^2&=y\end{aligned} \tag{15}$$

The top and bottom equations require that $x=0$ or 1 and $y=0$ or 1.

Now suppose that $x,y\in\{0,1\}$. From 10 and 13 we have 4 possibilities, shown in Table 1.

In all cases, we have $a*(b*c)=(a*b)*c$.

x	y	$xa+xyb+y^2c$	$x^2a+xyb+yc$
0	0	0	0
0	1	c	c
1	0	a	a
1	1	$a+b+c$	$a+b+c$

TABLE 1. Associative property.

We can also show that $*$ is commutative if and only if $x=y$. If $x=y$, then

$$a*b=xa+xb=x(a+b) \tag{16}$$

$$b*a=xb+xa=x(a+b) \tag{17}$$

If $*$ is commutative, then

$$a*b=xa+yb=b*a=xb+ya \tag{18}$$

Again, since a,b are arbitrary, we can equate coefficients, giving $x=y$.

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